Indian Statistical Institute
Second Semester 2006-2007
Mid Semestral Exam
B.Math III Year
Analysis IV

Time: 3 hrs | Date:05-03-07 | Max. Marks: 40

Answer all the questions:

1. Let (X, d) be a complete metric space. $f_1, f_2, \ldots : X \to \mathbb{C}$ a sequence of continuous functions such that for each x in X, $\sup\{|f_k(x)| : k = 1, 2, \ldots\} = C_x < \infty$. Show that there exists a nonempty open set V of X such that
$$\sup\{|f_k(x)| : k = 1, 2, \ldots \; x \in V\}$$
is finite. [3]

2. Let $g : R \to R$ be a C' function such that $g'(x_0) \neq 0$ for some x_0 in R. Let $y_0 = g(x_0)$. Show that there exists open sets U_0, V_0 such that $x_0 \in U_0, \quad y_0 \in V_0$, g is $1-1$ on $U_0, g(U_0) = V_0$, the inverse map $g^{-1} : V_0 \to U_0$ is continuous and differentiable. [5]

3. (a) Let $a_1, a_2 \ldots, a, \; b_1, b_2, \ldots, b$ be non negative real numbers. Let $a_n + b_n \to a + b$, $a \leq \liminf a_n$ and $b \leq \liminf b_n$. Show that $a_n \to a$ and $b_n \to b$. [2]
   (b) Let $f, f_1, f_2, \ldots : R \to [0, \infty)$ be integrable w.r.t Lebesgue measure. If $f_n \to f$ pointwise and $\int f_n \to \int f$, then show that $\int_E f_n \to \int_E f$ for each Borel subset E of R. [2]

4. Let $f : R \to \mathbb{C}$ be Lebesgue integrable. Define $g : R \to \mathbb{C}$ by
$$g(t) = \int f(x) e^{-itx} dx$$
   (a) Show that g is continuous. [1]
   (b) If further $\int |xf(x)| dx < \infty$, show that g is differentiable. [1]

5. Let $f, g : R \to R$ be Lebesgue integrable. Show that
$$\int f + g = \int f + \int g$$
[Recall:(i) The above equality is true if $f \geq 0$ and $g \geq 0$ and
(ii) $\int f = \int f^+ - \int f^-$] [3]

6. (a) Let $E_1 = \bigcup_{n=2}^{\infty} [n,\ n+1]$. Find $\lim_{y\to 0} \ell((E_1 + y\backslash E_1)\bigcup(E_1\backslash E_1 + y))$, where ℓ stands for Lebesgue measure and $E+y = \{x+y : \ x \in E\}$ [1]

(b) Let $E_2 = \bigcup_{n=2}^{\infty} [n,\ n+\frac{1}{n}]$. Find $\lim_{y\to 0} \ell[(E_2 + y\backslash E_2)\bigcup E_2\backslash(E_2 + y)]$ [2]

(c) Let $E_3 = \bigcup_{n=2}^{\infty} [n,\ n + a_n],\ 0 \le a_n \ \le \frac{1}{2} \ \sum_{2}^{\infty} a_n < \infty$ Find $\lim_{y\to 0} \ell((E_3 + y \ \backslash E_3)\bigcup(E_3\backslash E_3 + y))$ [1]

7. Let $\mathcal{B}$ be the Borel σ-algebra of R. Let $f : R \to R$ be a function such that $f^{-1}(a,\infty) \in \mathcal{B}$ for each a in R. Show that $f^{-1}(E) \in \mathcal{B}$ for each E in $\mathcal{B}$. [4]

8. Let $f : [0,\ 1] \to [0,\ \infty)$ be a Borel measurable bounded Riemann integrable function. Show that the Riemann integral of f and Lebesgue integral of f are qual. [4]

9. Let $f : R \to [0,\infty)$ be any bounded Borel measurable function. Show that there exists a sequence $0 \le s_1 \le \ s_2 \le s_3 \le \ldots$ of simple measurable functions such that $\lim_n \sup_x |s_n(x) - f(x)| = 0$. [3]

10. (a) Let $f : R \to R$ be given by

$$\begin{aligned} f(x) &= \frac{\sin x}{x} \quad \text{for } x \ge 1 \\ &= 0 \quad \text{if} \quad x < 1 \end{aligned}$$

Show that f is not Lebesgue integrable. [3]

(b) Show that $\lim_{k\to\infty} \int_1^k f(x)\ dx$ exists. [3]

11. Let $(X,\ d)$ be a complete metric space $F_1 \supset F_2 \supset \ldots$ a sequence of closed sets with dia $F_j \to 0$ as $j \to \infty$. Show that $\bigcap_1^{\infty} F_j$ is nonempty. [3]